PHY 108: Optical Physics

Solution to Homework #1

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Blackbody energy distribution is given by:

\[ \rho_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \left( \frac{1}{e^{\frac{h\nu}{kT}} - 1} \right) d\nu. \]  

(1)

**Problem #1**

Knowing that \( \nu \lambda = c \), rewrite the black body radiation formula in terms of wavelength.

**Solution:**

Since \( \nu \lambda = c \), we have \( \nu = \frac{c}{\lambda} \), and \( d\nu/d\lambda = -c/\lambda^2 \). Therefore, substitute these two formula into (1), we obtain:

\[ \rho_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \left( \frac{1}{e^{\frac{h\nu}{kT}} - 1} \right) d\nu, \]

\[ = -\frac{8\pi hc}{\lambda^5} \left( \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \right) d\lambda. \]

(2)

The minus sign reflects the fact that \( d\nu \) and \( d\lambda \) are of different sign. This is because, due to \( \nu \lambda = c \), if one of the two quantities \( \nu \) and \( \lambda \) increases, the other with decrease. That minus sign therefore guarantees the correctness of this process of changing variables. However, if we keep in mind this fact, and always interpret \( d\lambda \) positive, we can write:

\[ \rho_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \left( \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \right) d\lambda. \]

(3)

**Problem #2**

Total radiative flux from a blackbody radiator is often given as

\[ \Phi(T) = \sigma T^4, \]

where \( \sigma \) is called the Stephan-Boltzmann constant. Derive an expression for Stephan-Boltzmann Constant by evaluating the total radiative flux from a blackbody as a function of temperature by integrating the above expression as a function of frequency over a range from 0 to \( \infty \). (If the integration is done correctly, you will end up with an integral: \( \int_0^\infty \frac{x^3}{e^x - 1} dx \), whose numerical value is \( \frac{\pi^4}{15} \)).

**Solution:**

First, the energy density \( u \) (energy per volume) can be obtained by integrating
Figure 1: Illustration of computing the energy flux for blackbody radiation.

\[
\begin{align*}
(1) \text{ over } \nu \in [0, \infty): \\
& u = \int_0^\infty \rho_\nu \, d\nu = \int_0^\infty \frac{8\pi h\nu^3}{c^3} \left( \frac{1}{e^{\hbar\nu/kT} - 1} \right) \, d\nu \\
& x \equiv \frac{\hbar\nu}{kT} \rightarrow = \frac{8\pi k^4T^4}{c^3h^3} \int_0^\infty \frac{x^3}{e^x - 1} \, dx = \frac{8\pi^5k^4}{15c^3h^3}T^4,
\end{align*}
\]

where we used the identity:

\[
\int_0^\infty \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15}, \tag{5}
\]

which can be evaluated by Mathematica. It can be also worked out rigorously. The calculation can be found in almost any Statistical Physics books\(^1\), which requires the knowledge of Taylor expansion and Residue theorem.

For the radiative flux, it is essentially the energy flow through unit area per unit time. Consider a small area \(\Delta A\) and a small time interval \(\Delta t\), construct a spherical coordinate as in Fig. 1. If we construct a parallelepiped with a base of \(\Delta A\) and a side length of \(c\Delta t\), then all the energy inside this parallelepiped flowing toward \(\Delta A\) will contribute an amount of \(uc\Delta tA \cos \theta d\Omega/4\pi\) to the flux, where \(d\Omega = \sin \theta d\theta d\phi\) is the solid angle element along the direction of \((\theta, \phi)\). Therefore, the total flux can be obtained as:

\[
\Phi(T) = \frac{1}{\Delta A\Delta t} \int \frac{uc\Delta t \Delta A \cos \theta d\Omega/4\pi}{4\pi} \\
= \frac{uc}{4\pi} \int \cos \theta d\Omega = \frac{uc}{4\pi} \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi \\
= \frac{1}{4}cu. \tag{6}
\]

\(^1\)For example in the Appendix of F. Reif, "Fundamentals of Statistical and Thermal Physics" McGraw-Hill Book Company, 1965
Now substitute (4) into above equation, we get the Stefan-Boltzmann Law of:

$$\Phi(T) = \frac{2\pi^5 k^4}{15c^2 h^3} T^4 \equiv \sigma T^4,$$

where the Stefan-Boltzmann constant is:

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3} = 5.67 \times 10^{-8} \text{Wm}^{-2}\text{K}^{-4}. \quad (8)$$

**Problem #3**

Determine the temperature of a blackbody source that emits the 1mW of power in the same bandwidth and same divergence (solid angle) as a small He/Ne laser ($\lambda_0 = 632.8\text{nm}$, emitting area = $1\text{cm}^2$, divergence = $1\mu\text{rad}$ and $\Delta\nu = 10^9\text{Hz}$).

**Solution:**

For the blackbody with temperature $T$, the radiation energy in band $\nu \sim \nu + \Delta\nu$ is approximately $\rho_\nu \Delta\nu$. Now following a similar argument as in last problem, the power emitted through an area $\Delta A$ with a divergence of $\Delta\Omega$ is simply:

$$P = c \rho_\nu \Delta\nu \Delta A \frac{\Delta\Omega}{4\pi}. \quad (9)$$

Now according to the problem, $P = 1\text{mW} = 10^{-3}\text{W}$, $\Delta\Omega = 1\mu\text{rad} = 10^{-6}\text{rad}$, $\Delta A = 1\text{cm}^2 = 10^{-4}\text{m}^2$, $\nu = c/\lambda_0 = 4.74 \times 10^{14}\text{Hz}$, and $\Delta\nu = 10^9\text{Hz}$. We could solve from above equation:

$$\rho_\nu = \frac{4\pi P}{c \Delta\nu \Delta A \Delta\Omega} = 4.189 \times 10^{-7}\text{Js/m}^3, \quad (10)$$

and therefore:

$$T = 1.45 \times 10^{11}\text{K}. \quad (11)$$

**Problem #4**

Derive an expression for the peak of the Blackbody distribution as a function temperature by differentiating the expression (1) with respect to frequency and setting it equal to zero. You will come up with a transcendental equation to solve, which fortunately can be solved either graphically or numerically. For fun, not as a homework, try to evaluate the final answer.
Solution:

To find out the peak of the blackbody distribution, differentiate $\rho_\nu$ with respect to $\nu$, and set it equal to zero:

$$\frac{d\rho_\nu}{d\nu} = \frac{8\pi h}{c^3} \left[ \frac{3\nu^2}{e^{\hbar\nu/kT} - 1} - \frac{\hbar\nu^3}{kT} \left( e^{\hbar\nu/kT} - 1 \right)^2 \right] = 0.$$  \hspace{1cm} (12)

Let $x = h\nu/kT$, the above equation can be simplified into:

$$3(1 - e^{-x}) = x.$$  \hspace{1cm} (13)

The solution to this transcendental equation can be solved either numerically or graphically. For the graphical solution illustrated in Fig. 2, the curves $y = x$ and $y = 3(1 - e^{-x})$ intersects at $x_1 = 0$ and $x_2 \approx 2.82$, which are two solutions to the equation. The $x_1 = 0$ corresponds to a minimum of $\rho_\nu$ at $\nu = 0$, while the solution $x_2 \approx 2.82$ gives the position of the peak at:

$$\nu = \frac{xk}{h}T \approx \frac{2.82k}{h}T.$$  \hspace{1cm} (14)

To achieve a more accurate solution, we could solve the equation numerically by some algorithms, for example Newton’s method\(^2\). The result is $x_2 = 2.821439 \ldots$.

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\(^2\)See http://en.wikipedia.org/wiki/Newton’s_method