PHY 108: Optical Physics

Solution to Homework #3

TA: Xun Jia

May 5, 2008

\(^{1}\)Email: jiaxun@physics.ucla.edu
Consider the medium shown in the figure below:

![Medium with \( N_1 - N_2 < 0 \)](image)

\( l_{in} = 0 \) \hspace{1cm} \( l_{out} = ? \)

**Figure 1:** Illustration of the medium.

**Problem #1**

1 cm\(^2\) cross section, length = 1 meter, completely absorbing side walls, \( N_2 - N_1 = 0 \), and \( N_2 = 10^{10} \text{ cm}^{-3} \). Assume an atomic level separation corresponding to a wavelength of 500 nm.

(a). Calculate the solid angle in which the spontaneous emission emerges at \( l = 1 \) meter.

(b). Calculate the spontaneous emission power observed at the exit of the long tube at \( l = 1 \) meter.

**Solution:**

(a). The solid angle is simply:

\[
\Delta \Omega = \frac{A}{l^2} = \frac{1 \times 10^{-4}}{1^2} = 10^{-4} \text{ rad.} \tag{1}
\]

(b). Now consider a small volume of \( V = 1 \text{ cm}^3 \) at the left end of the medium. The total power emitted by the small volume is:

\[
I_{\text{total}} = h \nu A_{21} N_2 V = h \frac{c}{\lambda} A_{21} N_2 V \tag{2}
\]

\[
= 3.98 \times 10^{-2} \text{ W},
\]

where we take \( A_{21} = 10^7 \text{ sec}^{-1} \). Therefore, the power observed at the exit of the long tube is:

\[
I = \frac{\Delta \Omega}{4\pi} I_{\text{total}} = 3.16 \times 10^{-7} \text{ W.} \tag{3}
\]
Problem #2

Now consider the same geometry but the medium with population inversion, i.e., 1 cm$^2$ cross section, length = 1 meter, completely absorbing side walls $N_2 - N_1 > 0$, and $N_2 = 10^{10}$ cm$^{-3}$, and gain at linecenter = $g(0)$.

(a). Evaluate the expression for output linewidth narrowing assuming a naturally broadened lineshape.

(b). If medium exhibits a gain of $g(0) = 0.5$ cm$^{-1}$, calculate the total power emerging from the long tube at $l = 1$ meter.

Solution:

(a). The intensity of the incoming light as a function of frequency $\nu$ is:

$$I_{\text{in}}(\nu) = \frac{I_{\text{in}}(\nu_0)}{1 + \left[\frac{2(\nu - \nu_0)}{\Delta \nu_N}\right]^2},$$

where $\nu_0$ is the frequency at the band center, and the width of the band is $\Delta \nu_N$ assuming a naturally broadened lineshape. After the propagating through the gain medium for a distance of $l$, the outgoing light intensity varies as a function of $\nu$:

$$I_{\text{out}}(\nu) = I_{\text{in}}(\nu)e^{g(\nu)l},$$

where the gain is:

$$g(\nu) = \frac{g(\nu_0)}{1 + \left[\frac{2(\nu - \nu_0)}{\Delta \nu_N}\right]^2}.$$  

To calculate the width of the the outgoing light, we need to find the frequencies $\nu$ such that:

$$I_{\text{out}}(\nu) = \frac{1}{2}I_{\text{out}}(\nu_0).$$

Now, let $x = \left[\frac{2(\nu - \nu_0)}{\Delta \nu_N}\right]^2$, from the above equations we obtain:

$$\ln 2 = \ln(1 + x) + \frac{g(\nu_0)lx}{1 + x}.$$  

We are expecting the width for the outgoing light is much smaller than that for the incoming one, i.e., the solution $\nu$ to (7) are such that $\nu \sim \nu_0$. Therefore it is expected that $x \ll 1$. With this assumption, we could solve for $x$ from (8) and obtain:

$$x = \frac{\ln 2}{1 + g(\nu_0)l - \ln 2}.$$
and the two frequencies $\nu_1$ and $\nu_2$ which solve the equation (7) are therefore given by:

$$\nu_{1,2} = \nu_0 \pm \frac{1}{2}\sqrt{x}\Delta\nu_N.$$ \hspace{1cm} (10)

Finally, the width of the outgoing light is:

$$\Delta\nu_{\text{out}} = |\nu_1 - \nu_2| = \sqrt{x}\Delta\nu_N,$$

\hspace{1cm} (11)
i.e., the width of the outgoing light is reduced by $\sqrt{x}$ times as that of the incoming light. Normally, $x \ll 1$, and the width of the light is narrowed significantly after propagating through the gain medium.

(b). Again, consider a small volume $V = 1 \text{ cm}^3$ at the left end of the tube. The power emitted by it is given by (3). This amount of energy is amplified after propagating through the gain medium, and hence:

$$I_{\text{out}} = Ie^{g(\nu_0)l} = 1.64 \times 10^{15} \text{ W}.$$ \hspace{1cm} (12)