PHY 140A: Solid State Physics

Solution to Homework #6

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Problem #1

Debye frequency. A monatomic, cubic material has lattice spacing of $a$. The sound velocity for longitudinal and transverse phonons is approximately equal, $c_T = c_L = c$, is isotropic, and the highest phonon frequency is $\omega^*$. What is the Debye frequency? Does it depend on $\omega^*$?

Solution:

Debye frequency $\omega_D$ is a quantity introduced in Debye model to ensure that the total number of states in the model is correct. In Debye model, the dispersion relation is approximated to be $\omega = cK$, and in 3-D case the density of states is in the form:

$$D(\omega) = 3 \left( \frac{L}{2\pi} \right)^3 4\pi K^2 \frac{dK}{d\omega} = \frac{3V \omega^2}{2\pi^2 c^3} \quad (1)$$

where $V = L^3$, and the factor 3 is due to the fact that there are one longitudinal and two transversal vibration modes for each $K$ state. Therefore, if the total number of states in the solid is $N$, $\omega_D$ is determined by the following condition:

$$N = \int_{0}^{\omega_D} D(\omega) d\omega \quad (2)$$

In the cubic material, since there are three states associating with each atom, namely, two transversal and one longitudinal vibration, the total number of states $N = 3V/a^3$, so from above equation, we could solve:

$$\omega_D = \frac{c}{a} (6\pi^2)^{1/3} \quad (3)$$

On the other hand, $\omega^*$ is a real physical quantity: it is the highest phonon frequency, which is obtained by seeking a $K$ to maximize the dispersion relation for the normal modes. $\omega^*$ usually depends on the coupling constant $C$ in the solid and the mass of atoms $M$.

In general, there is no firm relationship between $\omega^*$ and $\omega_D$, however, they should be of the same order of magnitude.

Problem #2

Ashcroft and Mermin 1.1: Poisson Distribution. In the Drude model the probability of an electron suffering a collision in any infinitesimal interval $dt$ is just $dt/\tau$.

(a). Show that an electron picked at random at a given moment had no collision during the preceding t seconds with probability $e^{-t/\tau}$. Show that it will have no collision during the next t seconds with the same probability.

(b). Show that the probability that the time interval between successive collisions of an electron falls in the range between t and $t + dt$ is $(dt/\tau)e^{-t/\tau}$. 

1
(c). Show as a consequence of a) that at any moment the mean time back to the last collision (or up to the next collision) averaged over all electrons is $\tau$.

(d). Show as a consequence of b) that the mean time between successive collisions of an electron is $\tau$.

(e). (Optional) Part (c) implies that at any moment the time $T$ between the last and the next collision averaged over all electrons is $2\tau$. Explain why this is not inconsistent with the result in d). (A thorough explanation should include a derivation of the probability distribution for $T$.) A failure to appreciate this subtlety led Drude to a conductivity only half of the correct result $\sigma = ne^2\tau/m$.

**Solution:**

(a). We randomly pick an electron at a given moment, and divide the preceding time interval $t$ into $(t/dt)$ small intervals. In each part, with infinitesimal length $dt$, the probability that the electron suffers a collision is $dt/\tau$, thus the probability that it does not experience a collision in that $dt$ is $(1 - dt/\tau)$. This argument holds for every infinitesimal time interval, so the total probability that the electron does not have any collision in the total time $t$ will be the product of those probabilities in each infinitesimal interval:

$$P(t) = \lim_{dt \to 0} \left(1 - \frac{dt}{\tau}\right)^{t/dt} = \lim_{-dt/\tau \to 0} \left[\left(1 - \frac{dt}{\tau}\right)^{-\tau/dt}\right]^{-t/\tau} = e^{-t/\tau} \quad (4)$$

similarly, we can show that the electron will have no collision during the next $t$ seconds with the same probability.

(b). Consider the moment $t = 0$ right after a collision. Let $P_1$ be the probability that the electron has no collision in the finite time interval $[0, t]$, then $P_1 = e^{-t/\tau}$ due to a same argument as in part (a); let $P_2$ be the probability that the electron suffers a collision in time interval $[t, t + dt]$, then $P_2 = dt/\tau$ from Drude model assumption. Therefore, $P_1 \cdot P_2$ will be the probability that the electron experiences two successive collisions separated by time $t$, on the other word, the probability that the time interval between successive collisions of the electron falls in the range between $t$ and $t + dt$ is $P(t; dt) = P_1 \cdot P_2 = (dt/\tau)e^{-t/\tau}$.

(c). From part (a), among all $N$ electrons, at a given moment, on average, there will be roughly $Ne^{-t/\tau}$ electrons which experienced no collision in the preceding time $t$; and also, there will be roughly $Ne^{-(t+dt)/\tau}$ electrons which experienced no collision in the preceding time $(t+dt)$. Thus the number of electrons which have no collision in the preceding $t$, but suffer a collision in preceding interval $[t, t + dt]$ will be:

$$Ne^{-t/\tau} - Ne^{-(t+dt)/\tau} = -d[Ne^{-t/\tau}] = \frac{N}{\tau}e^{-t/\tau}dt \quad (5)$$
this is also the number of electrons for which the time interval back to last collision falls in \([t, t + dt]\). Average over all electrons, we get the mean time back to the last collision:

\[
\langle t \rangle = \frac{1}{N} \int_0^\infty t \frac{N dt}{\tau} e^{-t/\tau} = \tau
\]  

(6)

same argument is also true for the mean time up to the next collision.

(d). As a consequence of part (d), the mean time between two successive collision is simply computed as:

\[
\langle t \rangle = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} = \tau
\]  

(7)

Remark: Although part (c) and (d) are almost mathematically identical, the physical meaning is quite different. Part (c) is the average over whole electrons at a fix time, which is called "ensemble average"; while part (d) is the average over time for a given electron, which is called "time average". Even though the results are the same in those two ways of average in most of physical system, you should appreciate the physical meaning between them carefully.

(e). Let us consider the distribution of \(T\), the time interval between two successive collisions, as following. At \(t = 0\), if we randomly pick electrons, we will have probability \(e^{-t_1/\tau} dt_1/\tau\) to find those who will suffer the next collision in exactly time interval \([t_1, t_1 + dt_1]\); also, we will have probability \(e^{-t_2/\tau} dt_2/\tau\) to find those who had suffered a last collision in \([- (t_2 + dt_2), t_2]\). If we require the time interval between last and next collision is \(T\), then set \(T = t_1 + t_2\), we will have the probability of

\[
e^{-t_1/\tau} \frac{dt_1}{\tau} \cdot e^{-\left(T-t_1\right)/\tau} \frac{d\left(T-t_1\right)}{\tau} = e^{-T/\tau} \frac{dt_1}{\tau} \frac{dT}{\tau}
\]  

(8)

to pick those electrons who will suffer a next collision in \([t, t + dt]\), and the time interval between last and next collisions are \(T\).

Since we only care about the fact that the time interval between collision is \(T\), but not care when next collision happens, then by integrating over \(t_1\), we get the probability that the time interval between two successive collision lies in \(T, T + dT\):

\[
\int_0^T dt_1 e^{-T/\tau} \frac{1}{\tau} \frac{dT}{\tau} = e^{-T/\tau} \frac{T}{\tau} \frac{dT}{\tau}
\]  

(9)

note that the current moment lies in between last and next collision, that is why the upper limit of integral is \(T\) in above equation.

Finally, we could find out the average time interval based on the probability of (9):

\[
\langle T \rangle = \int_0^\infty T e^{-T/\tau} \frac{T}{\tau} \frac{dT}{\tau} = 2\tau
\]  

(10)
Problem #3

Ashcroft and Mermin 1.2: Joule Heating. Consider a metal at uniform temperature in a static uniform electric field $E$. an electron experiences a collision, and then, after a time $t$, a second collision. In the Drude model, energy is not conserved in collision, for the mean speed of an electron emerging from a collision does not depend on the energy that the electron acquired from the field since the time of the preceding collision.

(a). Show that the average energy lost to the ions in the second of two collisions separated by a time $t$ is $(eE\tau)^2/2m$. (The average is over all directions in which the electron emerged from the first collision.)

(b). Show, using the result of Problem 2b, that the average energy loss to the irons per electron per collision is $(ne^2\tau/m)E^2$, and hence that the average energy loss per cubic centimeter per second is $(ne^2\tau/m)E^2 = \sigma E^2$. Deduce that the power loss in a wire of length $L$ and cross section $A$ is $I^2R$, where $I$ is the current flowing and $R$ is the resistance of the wire.

Solution:

(a). Construct spherical coordinate $\theta$, and $\phi$ as in the figure. Consider an electron right after a collision with velocity $v$ as in the direction labeled by angle $(\theta, \phi)$. During the time interval $t$ between two successive collisions, the electron is accelerated by the electric field with an acceleration $a = Ee/m$. Then its velocity right before next collision will be $v_i = v + \Delta v = v + at = v + Eet/m$. After collision, the electron is scattered in to a certain direction randomly, however, the outgoing velocity satisfies $|v_f| = |v|$, which is determined by the temperature of the metal according to Drude mode.

![Figure 1: Illustration of the acceleration of electron in between two successive collisions.](image)
In this process, we could calculate the energy loss for that particular electron:

$$\Delta E(\theta, \phi) = \frac{1}{2} m(v_i^2 - v_f^2) = \frac{1}{2} m(v_i^2 - v^2)$$

$$= \frac{1}{2} m[(v^2 + \Delta v^2 + 2v\Delta v \cos \theta) - v^2] = \frac{(eEt)^2}{2m} + vEet \cos \theta$$

(11)

Since right after the first collision, the electron could be along any direction with equal probability, thus the probability from a small solid angle \(d\Omega\) around the direction \((\theta, \phi)\) is simply \(d\Omega/4\pi = \sin \theta d\theta d\phi/4\pi\). Averaging over all possible direction, we get the average energy lost to the ions in the second of two collisions separated by a time \(t\):

$$\Delta E = \frac{1}{4\pi} \int d\phi \int_0^{\pi} \sin \theta d\theta \Delta E(\theta, \phi) = \frac{(eEt)^2}{2m}$$

(12)

(b). From part (b) of previous problem, the probability that the time interval between successive collisions of an electron falls in the range between \(t\) and \(t + dt\) is \((dt/\tau)e^{-t/\tau}\), therefore, the average energy loss to the irons per electron per collision is:

$$\langle \Delta E \rangle = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} \Delta E = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} \frac{(eEt)^2}{2m} = \frac{(eE\tau)^2}{m}$$

(13)

Since, from last problem, the average time between two successive collisions is \(\tau\), the collision frequency will be \(1/\tau\), thus, the average energy loss per cubic centimeter per second, is the product of the energy loss for one electron per collision \(\langle \Delta E \rangle\), the density of electrons \(n\), and the collision frequency \(1/\tau\), i.e.,

$$\langle \Delta E \rangle \cdot n \cdot \frac{1}{\tau} = \frac{ne^2\tau}{m} E^2 \equiv \sigma E^2$$

(14)

Finally, the power loss will be the volume of the wire \(AL\) multiplied by the average energy loss per unit volume per unit time, \(\sigma E^2\):

$$P = AL \cdot \sigma E^2 = \sigma E^2 AL$$

(15)

moreover, from Drude model, \(j = \sigma E\), thus \(I = jA = \sigma EA\). Note that \(R = L/\sigma A\), so:

$$I^2R = \frac{(\sigma EA)^2 \cdot L}{\sigma A} = \sigma E^2 AL$$

(16)

compare above two equations, we conclude that power loss in the wire is \(I^2R\).

**Problem #4**

Kittel 6.6 **Drude model frequency dependence**. Use the equation \(m(dv/dt + v/\tau) = -eE\) for the electron drift velocity \(v\) to show that the conductivity at frequency \(\omega\) is

$$\sigma(\omega) = \sigma(0) \left( \frac{1 + i\omega \tau}{1 + (\omega \tau)^2} \right)$$
where $\sigma(0) = ne^2\tau/m$.

**Solution:**

For the equation $m\left(\frac{dv}{dt} + \frac{v}{\tau}\right) = -eE$, we are seeking for the solution in the form $v(t) = v_0 e^{-i\omega t}$ when the electric field is $E(t) = E_0 e^{-i\omega t}$. Substitute in the equation, we have:

$$mv_0 \left(-i\omega + \frac{1}{\tau}\right)e^{-i\omega t} = -eE_0 e^{-i\omega t}$$

which gives:

$$v_0 = \frac{-eE_0\tau}{m} \left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right)$$

therefore, the current density:

$$j = n(-e)v_0 = \frac{ne^2\tau}{m} \left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right) E_0$$

since $j = \sigma E$, we obtain the dependence of conductivity on $\omega$ as:

$$\sigma(\omega) = \sigma(0) \left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2}\right)$$

where $\sigma(0) = ne^2\tau/m$.

**Problem #5**

Study for the midterm.

**Solution:**

Good luck!