PHY 140A: Solid State Physics

Solution to Homework #7

Xun Jia

December 5, 2006

Email: jiaxun@physics.ucla.edu
**Problem #1**

**Static magnetoconductivity tensor.** For the drift velocity theory of (51), show that the static current density can be written in matrix form as:

\[
\begin{pmatrix}
 j_x \\
 j_y \\
 j_z
\end{pmatrix} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \begin{pmatrix}
 1 & -\omega_c \tau & 0 \\
 \omega_c \tau & 1 & 0 \\
 0 & 0 & 1 + (\omega_c \tau)^2
\end{pmatrix} \begin{pmatrix}
 E_x \\
 E_y \\
 E_z
\end{pmatrix}
\]  

(1)

In the high magnetic field limit of \( \omega_c \tau \gg 1 \), show that

\[
\sigma_{yx} = \frac{n e c}{B} = -\sigma_{xy}
\]

(2)

In this limit \( \sigma_{xx} = 0 \), to order \( 1/\omega_c \tau \). The quantity \( \sigma_{yx} \) is called the **Hall conductivity**.

**Solution:**

From Eqn. (51) in Kittel, in the static case, the velocity is not varying with time, so \( dv_i/dt = 0 \) for \( i = x, y, z \), therefore, in the matrix form, we have:

\[
\begin{pmatrix}
 m/\tau & eB/c & 0 \\
 -eB/c & m/\tau & 0 \\
 0 & 0 & m/\tau
\end{pmatrix} \begin{pmatrix}
 v_x \\
 v_y \\
 v_z
\end{pmatrix} = -e \begin{pmatrix}
 E_x \\
 E_y \\
 E_z
\end{pmatrix}
\]

(3)

solve this equation by inverting the coefficient matrix, and let \( \omega_c = eB/mc \), we get:

\[
\begin{pmatrix}
 v_x \\
 v_y \\
 v_z
\end{pmatrix} = \frac{-e\tau}{m[1 + (\omega_c \tau)^2]} \begin{pmatrix}
 1 & -\omega_c \tau & 0 \\
 \omega_c \tau & 1 & 0 \\
 0 & 0 & 1 + (\omega_c \tau)^2
\end{pmatrix} \begin{pmatrix}
 E_x \\
 E_y \\
 E_z
\end{pmatrix}
\]

(4)

since \( j = -ne v \) and \( \sigma_0 = ne^2 \tau/m \), it follows that:

\[
\begin{pmatrix}
 j_x \\
 j_y \\
 j_z
\end{pmatrix} = -ne \begin{pmatrix}
 v_x \\
 v_y \\
 v_z
\end{pmatrix}
\]

(5)

\[
\begin{pmatrix}
 j_x \\
 j_y \\
 j_z
\end{pmatrix} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \begin{pmatrix}
 1 & -\omega_c \tau & 0 \\
 \omega_c \tau & 1 & 0 \\
 0 & 0 & 1 + (\omega_c \tau)^2
\end{pmatrix} \begin{pmatrix}
 E_x \\
 E_y \\
 E_z
\end{pmatrix}
\]

In the high magnetic field limit, \( \omega_c \tau \gg 1 \), then:

\[
\sigma_{yx} = -\sigma_{xy} = \frac{\sigma_0 \omega_c \tau}{1 + (\omega_c \tau)^2} \approx \frac{\sigma_0 \omega_c \tau}{(\omega_c \tau)^2} = \frac{\sigma_0}{\omega_c \tau} = \frac{n e c}{B}
\]

(6)

and:

\[
\sigma_{xx} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \approx \frac{\sigma_0}{(\omega_c \tau)^2} = 0 \cdot \left( \frac{1}{\omega_c \tau} \right) + \frac{\sigma_0}{(\omega_c \tau)^2}
\]

(7)

thus \( \sigma_{xx} = 0 \), to order \( 1/\omega_c \tau \).
Problem #2

Kinetic energy of electron gas. Show that the kinetic energy of a three-dimensional gas of $N$ free electrons at $0K$ is

$$U_0 = \frac{3}{5} N\epsilon_F$$

(8)

Solution:

For the Fermi-Dirac distribution at $0K$, we have:

$$f(\epsilon) = \frac{1}{e^{(\epsilon - \epsilon_F)/k_BT} + 1} \bigg|_{T=0} = \begin{cases} 1 & \epsilon < \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases}$$

(9)

and the density of states of electrons in 3-D is:

$$D(\epsilon) = \frac{V}{(2\pi)^3} 4\pi k^2 \frac{dk}{d\epsilon} = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2}$$

(10)

since for free electrons, the kinetic energy equals to the total energy $\epsilon$, the average kinetic energy per electron is simply:

$$\bar{\epsilon} = \frac{\int_0^\infty d\epsilon D(\epsilon) f(\epsilon) \epsilon}{\int_0^\infty d\epsilon D(\epsilon) f(\epsilon)} = \frac{\int_0^{\epsilon_F} d\epsilon D(\epsilon) \epsilon}{\int_0^{\epsilon_F} d\epsilon D(\epsilon)} = \frac{3}{5} \epsilon_F$$

(11)

therefore the kinetic energy of total $N$ electrons is:

$$U_0 = N\bar{\epsilon} = \frac{3}{5} N\epsilon_F$$

(12)

Problem #3

Chemical potential in two dimensions. Show that the chemical potential of a Fermi gas in two dimensions is given by:

$$\mu(T) = k_BT \ln[\exp(\pi n\hbar^2/mk_BT) - 1]$$

(13)

for $n$ electrons per unit area. Note: the density of orbitals of a free electron gas in two dimensions is independent of energy: $D(\epsilon) = m/\pi\hbar^2$, per unit area of specimen.

Solution:

At finite temperature $T$, the chemical potential is determined by the condition:

$$n = \int_0^\infty d\epsilon D(\epsilon) f(\epsilon) = \int_0^{\epsilon_F} d\epsilon \frac{m}{\pi\hbar^2} \frac{1}{e^{(\epsilon - \mu)/k_BT} + 1}$$

(14)
the right hand side can be computed explicitly as:
\[
\int_0^\infty \frac{d\epsilon}{\pi \hbar^2} \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1} = \frac{m}{\pi \hbar^2} \int_0^\infty \frac{d\epsilon}{\pi \hbar^2} \frac{e^{-(\epsilon-\mu)/k_B T}}{1 + e^{-(\epsilon-\mu)/k_B T}}
\]
\[
= -\frac{mk_BT}{\pi \hbar^2} \int_0^\infty \frac{d(e^{-\epsilon/k_B T})}{1 + e^{\epsilon/k_B T}}
\]
\[
= -\frac{mk_BT}{\pi \hbar^2} \ln \left[ 1 + \exp \left( \frac{\epsilon}{k_B T} \right) \right]_{\epsilon=0}^\infty
\]
\[
= \frac{mk_BT}{\pi \hbar^2} \ln \left[ 1 + \exp \left( \frac{\mu}{k_B T} \right) \right]
\]
so from above two equations, we can solve for \(\mu\) as:
\[
\mu(T) = k_B T \ln \left[ \exp \left( \frac{\pi n \hbar^2}{m k_B T} \right) - 1 \right]
\]

Problem #4

Fermi gases in astrophysics.

(a). Given \(M_\odot = 2 \times 10^{33} \text{g}\) for the mass of the Sun, estimate the number of electrons in the Sun. In a white dwarf star this number of electrons may be ionized and contained in a sphere of radius \(2 \times 10^9 \text{cm}\); find the Fermi energy of the electrons in electron volts.

(b). The energy of an electron in the relativistic limit \(\epsilon \gg mc^2\) is related to the wavevector as \(\epsilon \simeq \hbar c k = \hbar c k\). Show that the Fermi energy in this limit is \(\epsilon_F \approx \hbar c (N/V)^{1/3}\), roughly.

(c). If the above number of electrons were contained within a pulsar of radius \(10 \text{km}\), show that the Fermi energy would be \(\approx 10^8 \text{eV}\). This explains why pulsars are believed to be composed largely of neutrons rather than of protons and electrons, for the energy release in the reaction \(n \rightarrow p + e^-\) is only \(0.8 \times 10^6 \text{eV}\), which is not large enough to enable many electrons to form a Fermi sea. The neutron decay proceeds only until the electron concentration builds up enough to create a Fermi level of \(0.8 \times 10^6 \text{eV}\), at which point the neutron, proton, and electron concentrations are in equilibrium.

Solution:

(a). The Sun is mainly composed by hydrogen atoms, whose mass is approximately equal to the mass of proton \(m_p\). Since each hydrogen atom contains one electron, the number of electrons in the Sun is:
\[
N = \frac{M_\odot}{m_p} = 1.2 \times 10^{57}
\]
if this number of electrons is contained in a white dwarf star with a radius $R_{wd} = 2 \times 10^9$ cm, the Fermi energy is:

$$
\epsilon_F = \frac{\hbar^2}{2m} \left( 3\pi^2 N \frac{V}{V} \right)^{2/3} = \frac{\hbar^2}{2m} \left( 3\pi^2 \frac{N}{4\pi R_{wd}^3} \right)^{2/3}
$$

(18)

$$
= 6.3 \times 10^{-15} J = 4 \times 10^4 eV
$$

(b). In the relativistic case with a dispersion relation of $\epsilon \simeq pc = \hbar kc$, the density of states of electrons is:

$$
D(\epsilon) = 2 \frac{V}{(2\pi)^3} \frac{4\pi k^2}{\epsilon^2} d\epsilon = \frac{V \epsilon^2}{\pi^2 \hbar^3 c^3}
$$

(19)

then the Fermi energy is determined by the condition:

$$
\epsilon_F = \int_0^{\epsilon_F} D(\epsilon) d\epsilon = \frac{V \epsilon_F^3}{3\pi^2 \hbar^3 c^3}
$$

(20)

which gives:

$$
\epsilon_F = \hbar c \left( 3\pi^2 N \frac{V}{V} \right)^{1/3} \sim \hbar c \left( \frac{N}{V} \right)^{1/3}
$$

(21)

(c). If above number of electrons is contained in a pulsar of radius $R_p = 10km$, the Fermi would be:

$$
\epsilon_F = \hbar c \left( \frac{N}{V} \right)^{1/3} = \hbar c \left( \frac{N}{4\pi R_p^3} \right)^{1/3} = 2.1 \times 10^{-11} J = 1.3 \times 10^8 eV
$$

(22)

### Problem #5

**Liquid $He^3$.** The atom $He^3$ has spin $\frac{1}{2}$ and is a fermion. The density of liquid $He^3$ is $0.081 g/cm^3$ near absolute zero. Calculate the Fermi energy $\epsilon_F$ and the Fermi temperature $T_F$.

**Solution:**

The mass of a single $He^3$ is about:

$$
m_{He} = 3 \text{ a.m.u.} = 3 \times 1.67 \times 10^{-27} kg = 5.01 \times 10^{-27} kg
$$

(23)

therefore the number density of $He^3$ is

$$
n = \frac{\rho}{m_{He}} = 1.62 \times 10^{28} m^{-3}
$$

(24)
so the Fermi energy is:

\[
\epsilon_F = \frac{\hbar^2}{2m_{He}} \left(\frac{3\pi^2 N}{V}\right)^{2/3} = \epsilon_F = \frac{\hbar^2}{2m_{He}} (3\pi^2 n)^{2/3} = 6.74 \times 10^{-23} \text{J} = 4.2 \times 10^{-4} \text{eV}
\]  

(25)

and the Fermi temperature is:

\[
T_F = \frac{\epsilon_F}{k_B} = 4.88 \text{K}
\]  

(26)